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ON THE FORM OF THE POWER SERIES FOR AN ALGEBRAIC FUNCTION.

By E. J. WILCZYNSKI, University of Chicago.

It is a very familiar fact that the expansion of a rational function into a power series is characterized by the property that its coefficients are connected by a so-called scale of relation. In other words: if $w = f(z)$ is a rational function of z, and if w can be expanded into a power series of the form

$$w = f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_k z^k + \cdots,$$

there will exist a positive integer n, such that for all values of $k \geqq n$, a relation of the form

$$a_k = A_1 a_{k-1} + A_2 a_{k-2} + \cdots + A_p a_{k-p}, \qquad (k = n, n+1, n+2, \cdots),$$

will hold between $p + 1$ consecutive coefficients of the expansion, the quantities $A_1, A_2, \cdots, A_p$ being constants independent of k. Conversely any expansion of this form defines w as a rational function of z.

Thus, the existence of a scale of relation is a necessary and sufficient condition for a power-series which defines a rational function. The corresponding condition for the power series expansion of a general algebraic function seems to have escaped notice, although it may be obtained by the following simple and elementary process.

Let us consider the algebraic function w, of z, which is defined by the algebraic equation

$$(1) \qquad F(z, w) = \varphi_0(z) w^n + \varphi_1(z) w^{n-1} + \cdots + \varphi_{n-1}(z) w + \varphi_n(z) = 0,$$

where the coefficients $\varphi_k(z)$ are polynomials of degree no higher than m, so that we may write

$$(2) \qquad \varphi_k(z) = A_0^{(k)} + A_1^{(k)} z + \cdots + A_m^{(k)} z^m,$$

where the quantities $A_i^{(k)}$ are constants.

Let us suppose that one of the roots of (1) may be expanded as a power series in the neighborhood of $z = 0$, and let

$$(3) \qquad w = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{i=0}^{\infty} a_i z^i$$

be this power series. From (3) we may obtain the expansions of $w^2, w^3, \cdots, w^n$ by the familiar process of multiplication of power-series. Let

$$(4) \qquad w^k = \sum_{i=0}^{\infty} a_i^{(k)} z^i, \qquad (k = 0, 1, 2, \cdots, n),$$

be the expansions obtained in this way.

If (3) actually represents a root of (1), substitution of (4) in (1) must give rise to an identity; that is, the total coefficient of each power of z must be equal to zero. Let us make this substitution, and equate to zero the coefficient of z^k. For all values of $k \leqq m$, we find

$$
\begin{aligned}
&(A_0{}^{(0)}a_k{}^{(n)} + \cdots + A_m{}^{(0)}a_{k-m}{}^{(n)}) + (A_0{}^{(1)}a_k{}^{(n-1)} + \cdots + A_m{}^{(1)}a_{k-m}{}^{(n-1)}) \\
&\qquad + \cdots + (A_0{}^{(i)}a_k{}^{(n-i)} + \cdots + A_m{}^{(i)}a_{k-m}{}^{(n-i)}) \\
(5)\qquad &\qquad + \cdots + (A_0{}^{(n-1)}a_k + \cdots + A_m{}^{(n-1)}a_{k-m}) \\
&\qquad + A_k{}^{(n)} = 0, \qquad\qquad (k \leqq m),
\end{aligned}
$$

and for all higher values of k,

$$
\begin{aligned}
&(A_0{}^{(0)}a_k{}^{(n)} + \cdots + A_m{}^{(0)}a_{k-m}{}^{(n)}) + (A_0{}^{(1)}a_k{}^{(n-1)} + \cdots + A_m{}^{(1)}a_{k-m}{}^{(n-1)}) \\
&\qquad + \cdots + (A_0{}^{(i)}a_k{}^{(n-i)} + \cdots + A_m{}^{(i)}a_{k-m}{}^{(n-i)}) \\
(6)\qquad &\qquad + \cdots + (A_0{}^{(n-1)}a_k + \cdots + A_m{}^{(n-1)}a_{k-m}) = 0, \\
&\qquad\qquad (k = m + 1, m + 2, m + 3, \cdots).
\end{aligned}
$$

If $n = 1$, equation (1) defines w as a rational function of z, and the relations (6) reduce to the familiar scale of relation. In the general case, relation (6) replaces the scale of relation. As in the scale of relation, the coefficients $A_i{}^{(j)}$ which occur in (6) are independent of k. In fact they are merely the constant coefficients which appear in the defining algebraic equation (1). But the relation (6) is *not* in general, a *linear* relation between the coefficients of the power series. It *is* linear in the $n(m + 1)$ quantities $a_k, \cdots a_{k-m}, a_k{}^{(2)}, \cdots, a_{k-m}{}^{(2)}, \cdots, a_k{}^{(n)}, \cdots, a_{k-m}{}^{(n)}$, but it is of degree n with respect to the quantities $a_1, \cdots, a_k$. We shall therefore speak of a relation of form (6) as *a scale of relation of degree n*. Such a scale is of the same form as the familiar simplest case except that it is concerned not only with the coefficients of the expansion of w itself, but also with the coefficients of the expansion of $w^2, w^3, \cdots, w^n$. We may summarize our result as follows, including an obvious generalization which consists in replacing z by $z - \alpha$.

If w is an algebraic function of z, defined by an equation of degree n in w and of degree m in z, and if w can be expanded in a convergent series proceeding according to positive integral powers of $z - \alpha$, then the coefficients of this expansion satisfy a scale of relation of degree n, which will contain at most $(m + 1)n$ terms.

The converse is immediate. If the coefficients a_k of a power series satisfy a given scale of relation of degree n, the coefficients, $A_i{}^{(j)}$, $(j = 0, 1, \cdots, n - 1;$ $i = 0, 1, \cdots, m)$, of the scale will determine the coefficients $\varphi_0(z), \varphi_1(z), \cdots,$ $\varphi_{n-1}(z)$ of an algebraic equation of form (1) between w and z The constant coefficients of $\varphi_n(z)$ may then be determined by means of (5). These latter coefficients depend not only upon the given scale of relation but also upon the values of the first $m + 1$ coefficients of the given power series. The algebraic equation between w and z, obtained in this way, will have as one of its roots the function with the given expansion.

Thus, *any convergent power-series, whose coefficients satisfy a scale of relation of degree n, defines an algebraic function which is a root of an equation of degree n in w, and whose coefficients are integral rational functions of z.*

Of course we are concerned only with the form of the power series expansion for w whenever such an expansion *exists.* We have not discussed the exceptional cases when such an expansion is impossible. But these exceptional cases are well known. Thus if α is a zero of $\varphi_0(z)$ the expansion of w in powers of $z - \alpha$ may contain negative powers. If α is a root of the equation obtained by eliminating w from $F(z, w) = 0$ and $\partial F/\partial w = 0$, there may be fractional powers of $z - \alpha$ in such an expansion. If $z = \alpha$ is a value of z different from these critical values, an expansion in powers of $z - \alpha$ involving only positive integral powers is known to exist, and the coefficients of the expansion will obey the law which we have indicated. As a matter of fact such expansions frequently exist even in the neighborhood of the critical values. Moreover it is easy to make the modifications necessary to describe the form of the expansion when fractional or negative powers occur in it. Finally it may be stated that the critical values of z might also be rediscovered by investigating directly those cases in which the series obtained from (5) and (6) fail to converge.

The equation (1) may be reducible, that is, it may be possible to express its left member as a product of two integral rational functions of lower degree. In that case we shall still have a scale of relation of form (6) but with smaller values for m or n or both.

The scale of relation for the coefficients of the power-series expansion of an algebraic function will actually be a scale of degree n (and not lower), if the equation (1) *which defines this algebraic function is an irreducible equation of degree n in w.*

We have found a necessary and sufficient condition in order that the function w defined by the given power-series may be algebraic. We may express this condition in another way.

Let us write the matrix of the coefficients of $A_0^{(0)} \cdots A_m^{(n-1)}$, in equations (6), that is, the matrix of mn columns and infinitely many rows:

$$(7)\qquad \begin{array}{l}
a_{m+1}, \cdots, a_1, a_{m+1}^{(2)}, \cdots, a_1^{(2)}, \cdots, a_{m+1}^{(n)}, \cdots, a_1^{(n)}, \\
a_{m+2}, \cdots, a_2, a_{m+2}^{(2)}, \cdots, a_2^{(2)}, \cdots, a_{m+2}^{(n)}, \cdots, a_2^{(n)}, \\
a_{m+3}, \cdots, a_3, a_{m+3}^{(2)}, \cdots, a_3^{(2)}, \cdots, a_{m+3}^{(n)}, \cdots, a_3^{(n)}, \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
a_{m+k}, \cdots, a_k, a_{m+k}^{(2)}, \cdots, a_k^{(2)}, \cdots, a_{m+k}^{(n)}, \cdots, a_k^{(n)}, \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad .
\end{array}$$

Clearly, the existence of relations of the form (6) is equivalent to the vanishing of all of the determinants of order mn of this matrix. Therefore we obtain the following result:

If the function defined by the power series (3) *is an algebraic function there must exist two finite positive integers, m and n, such that all of the determinants of order mn of the matrix* (7) *will vanish. Conversely if, for given values of m and n, all of the determinants of order mn of the matrix* (7) *do vanish, then w will be a root of an algebraic equation of degree m in z and of degree n in w.*

It is clear that the systematic investigation of all such determinants will disclose the lowest values of m and n which will serve for a given power series, and that we may then actually determine the irreducible algebraic equation which the given power-series will actually satisfy. If no finite values of m and n can be found which satisfy the conditions of the theorem, the given power-series defines a transcendental function.

The remarkable theorem of EISENSTEIN,[1] which furnishes a necessary condition on the coefficients a_k in order that w may be algebraic, in the special case when the coefficients a_k are all rational numbers, may be regarded as a consequence of the general theorems stated in this paper.

CONCERNING A METHOD FOR FINDING A PARTICULAR INTEGRAL.[2]

By ARTHUR B. COBLE, University of Illinois.

The problem considered here is the determination of a particular integral U of the linear differential equation with constant coefficients,

$$(1) \qquad f(D)y \equiv (k_0D^n + k_1D^{n-1} + \cdots + k_n)y = X\left(D = \frac{d}{dx}\right),$$

in the special case when the right-hand member X is the sum of a number of terms $g_1(x), \cdots, g_j(x)$ such that these terms and all of the terms which arise from them by differentiation can be expressed linearly with constant coefficients by means of a finite number

$$(2) \qquad g_1(x), g_2(x), \cdots, g_l(x)$$

of the terms. Such terms for example are x^ρ, $e^{\alpha x}$, $\sin^\rho \beta x$, $\cos^\rho \gamma x$, $\sinh^\rho \delta x$, $\cosh^\rho \epsilon x$ (ρ any positive integer), and any products of these.

As is customary we call $f(m) = 0$ the *auxiliary equation;* and the complete solution of the differential equation $f(D)y = 0$, the *complementary function* of (1). Each r-fold root, $m = \alpha$, of $f(m) = 0$ contributes a part

$$(3) \qquad G(\alpha, r) = (\lambda_1x^{r-1} + \lambda_2x^{r-2} + \cdots + \lambda_r)e^{\alpha x}$$

to the complementary function.

The rule for the determination of the particular integral U of (1) can be stated as follows:

I. *Substitute for y in* (1) *the trial integral $U = c_1g_1(x) + c_2g_2(x) + \cdots + c_lg_l(x)$ and determine the coefficients $c_1, \cdots, c_l$ by equating coefficients of $g_1, \cdots, g_l$. If*

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[1] Stated by him without proof. For a proof consult Heine, "Der Eisenstein'sche Satz über Reihenentwickelungen aller algebraischer Funktionen," *Crelle's Journal*, Vol. 45 (1853), pp. 285–382.

[2] Read before the Md.-Va.-D. C. Section of the Mathematical Association, at Annapolis, December 15, 1917.